POLES OF THE TOPOLOGICAL ZETA FUNCTION ASSOCIATED TO AN IDEAL IN DIMENSION TWO

Lise Van Proeyen and Willem Veys [*]

Abstract

To an ideal in $\mathbb{C}[x, y]$ one can associate a topological zeta function. This is an extension of the topological zeta function associated to one polynomial. But in this case we use a principalization of the ideal instead of an embedded resolution of the curve.

In this paper we will study two questions about the poles of this zeta function. First, we will give a criterion to determine whether or not a candidate pole is a pole. It turns out that we can know this immediately by looking at the intersection diagram of the principalization, together with the numerical data of the exceptional curves. Afterwards we will completely describe the set of rational numbers that can occur as poles of a topological zeta function associated to an ideal in dimension two. The same results are valid for related zeta functions, as for instance the motivic zeta function.

1 Introduction

We will first define the topological zeta function for one polynomial in n variables over $\mathbb{C}$ and mention a number of important results about the poles of these functions. Afterwards, we will concentrate on the topological zeta function associated to an ideal in $\mathbb{C}[x, y]$ and make some similar statements about its poles.

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a non-constant polynomial satisfying $f(0) = 0$. To define the topological zeta function $Z_{top,f}(s)$, we take an embedded resolution $h : X \to \mathbb{C}^n$ of $f^{-1}\{0\}$. Let E_i for $i \in S$ be the irreducible components of $h^{-1}(f^{-1}\{0\})$, then we denote by N_i and $\nu_i - 1$ the multiplicities of E_i in the divisor on X of $f \circ h$ and $h^*(dx_1 \wedge \ldots \wedge dx_n)$, respectively. (Further on we

give a description of these multiplicities with local coordinates.) With these numerical data we can define the local *topological zeta function associated to f* :

$$Z_{top,f}(s) := \sum_{I \subset S} \chi(E_I^\circ \cap h^{-1}\{0\}) \prod_{i \in I} \frac{1}{N_i s + \nu_i},$$

where $\chi(\cdot)$ denotes the topological Euler-Poincaré characteristic and $E_I^\circ := (\cap_{i \in I} E_i) \backslash (\cup_{j \notin I} E_j)$.

There is also a *global* topological zeta function, where we replace $E_I^\circ \cap h^{-1}\{0\}$ by E_I°. Denef and Loeser proved in [DL1] that these definitions are independent of the choice of the resolution.

In particular, the poles of the topological zeta function of f are interesting numerical invariants. For example, the monodromy conjecture relates the poles with eigenvalues of the local monodromy of f (see e.g. [DL1]). It is easy to see that all poles belong to the set $\{-\nu_i/N_i \,|\, i \in S\}$. These elements are called the candidate poles associated to the given resolution. They are all negative rational numbers. It is an important question to determine whether or not a candidate pole is a pole.

In [V3], the second author proved a fast criterion to answer this question if we work with a curve $f \in \mathbb{C}[x_1, x_2]$. He showed that we can read the poles out of the minimal embedded resolution of the curve: a candidate pole s_0 is a pole if and only if $s_0 = -\frac{\nu_i}{N_i}$ for some exceptional curve E_i intersecting at least three times other components or $s_0 = -\frac{1}{N_i}$ for some irreducible component E_i of the strict transform of f.

There are also various results about the set

$$\mathcal{P}_n := \{s_0 \,|\, \exists f \in \mathbb{C}[x_1, \ldots, x_n] : Z_{top,f}(s) \text{ has a pole in } s_0\}.$$

For example, in [LSV] it is shown that each rational number in the interval $[-(n-1)/2, 0)$ is contained in $\mathcal{P}_n$. For $n = 2$ this means that we know $\mathcal{P}_2$ completely, as in [SV] it is proven that $\mathcal{P}_2 \cap (-\infty, -1/2\,) = \{-1/2 - 1/i \,|\, i \in \mathbb{Z}_{>1}\}$.

The construction of blowing-up that is used to desingularize varieties, can also be used to principalize an ideal. This means that after these blow-ups, the ideal is locally principal and monomial. This is a result of Hironaka [H].

Theorem 1.1. (Hironaka.) *Let X_0 be a smooth algebraic variety over a field of characteristic zero, and $\mathcal{I}$ a sheaf of ideals on X_0. There exists a principalization of $\mathcal{I}$, that is a sequence*

$$X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} X_2 \cdots \xleftarrow{\sigma_i} X_i \longleftarrow \cdots \xleftarrow{\sigma_r} X_r = X$$

of blow-ups $\sigma_i : X_{i-1} \leftarrow X_i$ in smooth centers $C_{i-1} \subset X_{i-1}$ such that

2

1. *the exceptional divisor E^i of the induced morphism $\sigma^i = \sigma_1 \circ \ldots \circ \sigma_i : X_i \to X_0$ has only simple normal crossings and C_i has simple normal crossings with E^i, and*

2. *the total transform $(\sigma^r)^*(\mathcal{I})$ is the ideal of a simple normal crossings divisor E. If the subscheme determined by $\mathcal{I}$ has no components of codimension one, then E is a natural combination of the irreducible components of the divisor E^r.*

Remark 1.2. In order to denote the total transform $(\sigma^r)^*(\mathcal{I})$, other authors may use the notation $\mathcal{I}\mathcal{O}_X$. If $\mathcal{I}$ has components of codimension one, we can write the total transform as a product of two (principal) ideals: the support of the first one is the exceptional locus, where the support of the second one is formed by the irreducible components of the total transform that are not contained in the exceptional locus. This second ideal is the 'weak transform' of $\mathcal{I}$.

When we have a principalization $\sigma = \sigma^r$, we can define numerical data (N, ν) for each component of the support of $\sigma^*(\mathcal{I})$ such that for every $b \in X$ there exist local coordinates $(y_1, \ldots, y_n)$ which satisfy the following conditions:

- if $E_1, \ldots, E_p$ are the irreducible components of the divisor E containing b, we have on some neighbourhood of b that E_i is given by $y_i = 0$ for $i = 1, \ldots, p$,

- $\sigma^*(\mathcal{I})$ is generated by $\varepsilon(y) \prod_{i=1}^{p} y_i^{N_i}$, and

- $\sigma^*(dx_1 \wedge \ldots \wedge dx_n) = \eta(y) \prod_{i=1}^{p} y_i^{\nu_i - 1} dy_1 \wedge \ldots \wedge dy_n$,

where $\varepsilon(y)$ and $\eta(y)$ are units in the local ring of X at b.

We can associate a topological zeta function to an ideal $\mathbf{f} = (f_1, \ldots, f_l)$, where we suppose that $0 \in \operatorname{Supp}(\mathbf{f})$. We use the numerical data that originate from a chosen principalization to define the local *topological zeta function*

$$Z_{top, \mathbf{f}}(s) := \sum_{I \subset T} \chi(E_I^\circ \cap \sigma^{-1}(0)) \prod_{i \in I} \frac{1}{\nu_i + sN_i},$$

with $E_i(N_i, \nu_i)$ for $i \in T$ the components of the support of the total transform of $\mathbf{f}$, and again $E_I^\circ = (\cap_{i \in I} E_i) \backslash (\cup_{j \notin I} E_j)$.

When $l = 1$, Denef and Loeser showed in [DL1] that the expression above does not depend on the chosen resolution by expressing it as a limit of p-adic Igusa zeta functions. They introduced later in [DL2], still for $l = 1$, the motivic zeta function of f, which is intrinsically defined. It has however a formula of the same kind as above in terms of a resolution. Specializing this formula to Euler characteristics yields the topological zeta function of f.

One can associate more generally a motivic zeta function to an ideal and obtain a similar formula in terms of a principalization using the same argument as in [DL2]. Again specializing to Euler characteristics yields the defining expression above for the topological zeta function of an ideal. This generalization to ideals is mentioned in [VZ, (2.4)].

Alternatively, one can check that this expression is independent of the chosen principalization by verifying that it is invariant under a blow-up with allowed center (this is straightforward) and then applying the Weak Factorization Theorem of Włodarczyk et al. [AKMW]. Note that in dimension 2 one does not need the Weak Factorization Theorem since there is a minimal principalization.

Remark 1.3. As in the case of one polynomial, there is also a *global* version of this zeta function, where we replace $E_I^\circ \cap \sigma^{-1}(0)$ by E_I°. However, in this paper we will work with the local one.

Now we can ask the same questions for the topological zeta function of an ideal in $\mathbb{C}[x,y]$ as we mentioned for the case of one polynomial: how can we determine which candidate poles are poles? Which rational numbers occur as poles of a zeta function of an ideal in dimension two?

Theorem 4.2 will answer the first question as a generalization of the result of the second author for the topological zeta function of a curve. It turns out that you can determine which candidate poles are poles by drawing an intersection diagram of the E_i associated to the minimal principalization *together with their numerical data.* In the case of one polynomial, a component of the strict transform as well as an exceptional variety that intersects at least three times an other component, give rise to a pole. This will still hold for the topological zeta function of an ideal in dimension two. But this time it is not true that an exceptional variety that intersects once or twice an other component never causes a pole. Sometimes it will, sometimes it won't. To solve this question, we will associate a "generic" curve to the ideal and we will prove that a principalization of the ideal also gives an embedded resolution of this curve with the same numerical data. Afterwards, we show how these numerical data tell us whether or not a candidate pole is a pole in this case.

Further on in this paper we will answer the second question. We will show that the possible poles of a zeta function of an ideal in dimension two, are exactly the rational numbers in $[-1,0) \cup \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>0}\}$ (see Theorem 5.3).

In the end, we will also draw conclusions about poles of other zeta functions of ideals in dimension two. In fact, we can say that the same results as we prove for the topological zeta function, are also true for the Hodge and the motivic zeta function and for most p-adic Igusa zeta functions. We don't need to prove these statements separately, but we can extract them out of the results for the topological zeta function.

2 Resolution of a generic curve

Let $\mathbf{f} = (f_1, \dots, f_l)$ be an ideal in $\mathbb{C}[x, y]$. We suppose in this section that $l > 1$. Then we can look at the linear system $\{\lambda_1 f_1 + \dots + \lambda_l f_l \mid \lambda_i \in \mathbb{C}$ for $i = 1, \dots, l\}$. A generic curve of $\mathbf{f}$ is a general element of this linear system. So actually, the definition of a generic curve of an ideal is dependent on the generators we use to represent the ideal.

Lemma 2.1. *A series of blow-ups used to principalize an ideal of $\mathbb{C}[x, y]$, also gives an embedded resolution of a generic curve of this ideal.*

Remark 2.2. This resolution will -in general- not be minimal, but we can still use a lot of the results about the numerical data of an embedded resolution and use them in our context.

Proof. When we start with an ideal $\mathcal{I} = (f_1, \dots, f_l) \subset \mathbb{C}[x, y]$, we can first determine whether there are common components among the f_i and put them together. So we will write

$$\mathcal{I} = (h)(f_1', \dots, f_l')$$

with $(f_1', \dots, f_l')$ a finitely supported ideal.

We need two chains of blow-ups to have a principalization:

(A) a composition of blow-ups $\sigma : \tilde{X} \to \mathbb{C}^2$ to transform $(f_1', \dots, f_l')$ in a locally principal ideal, and

(B) a series of blow-ups $\tau : X \to \tilde{X}$ to desingularize the strict transform of $h = 0$ and make it have normal crossings with all exceptional curves.

We will look now at the situation after the first series of blow-ups. The ideal $\sigma^* \mathcal{I} = (f_1^*, \dots, f_l^*)$ is locally principal. So in every point $b \in \tilde{X}$ we have local coordinates $(y_1, \dots, y_n)$ and a generator $g(y)$ such that

$$f_i^*(y) = g(y)\tilde{f}_i(y)$$

for $i = 1, \dots, l$. Moreover, we know that there exist regular functions $\mu_i(y)$ on $\tilde{X}$ to write that $g(y) = \sum_{i=1}^{l} \mu_i(y) f_i^*(y)$. So $g(y) = g(y) \sum_{i=1}^{l} \mu_i(y)\tilde{f}_i(y)$ and $1 = \sum_{i=1}^{l} \mu_i(y)\tilde{f}_i(y)$. We can conclude that the $\tilde{f}_i(y)$ don't have a common zero.

We study the linear system $\{\lambda_1 \tilde{f}_1 + \dots + \lambda_l \tilde{f}_l = 0 \mid \lambda_i \in \mathbb{C}$ for $i = 1, \dots, l\}$. This is a linear system without base points. By Bertini's theorem (see e.g. [J, Theorem 6.10]) we know that a general element of the system is non-singular and connected.

We can also restrict the linear system to an exceptional curve or to a component of the strict transform of $h = 0$. (Note that there are a finite number of such varieties.) On these curves, we get a new linear system

without base points. We can again use the theorem of Bertini to say that a general element is non-singular. In this case, this means that every intersection point of a general element of the original linear system with a component of the strict transform of $h = 0$ or with an exceptional curve has intersection multiplicity one.

Now we look at the following set of points: intersection points of an exceptional curve with a component of a strict transform of $h = 0$ and singular points of the strict transform of $h = 0$. This is a finite set. A general element of the linear system doesn't contain any of them.

We use all this to conclude the following: if we take a generic curve $\lambda_1 f_1' + \ldots + \lambda_l f_l' = 0$ with $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ (necessarily reduced by Bertini's Theorem), we can suppose that the strict transform of this curve after the first series of blow-ups (which is locally given by $\lambda_1 \tilde{f}_1 + \ldots + \lambda_l \tilde{f}_l = 0$) is non-singular, intersects the strict transform of $h = 0$ and the exceptional curves transversely, and doesn't contain any of the points in the mentioned set. This implies that after series (B), the components of the strict transform of the generic curve $\lambda_1 f_1 + \ldots + \lambda_l f_l = 0$ are still non-singular and the transform $(\sigma \circ \tau)^*(\lambda_1 f_1 + \ldots + \lambda_l f_l) = 0$ is a normal crossings divisor. So a principalization of the ideal $(f_1, \ldots, f_l)$ gives also an (in general non-minimal) embedded resolution of a generic curve $\lambda_1 f_1 + \ldots + \lambda_l f_l = 0$. $\square$

Remark 2.3. This lemma is well-known. We stated and proved it in dimension two, but one can do the same in higher dimensions. In our proof, we made a separation in two series of blow-ups. This is not really necessary and in higher dimensions one better avoids this. However, we chose to make this break to get a clearer view on the role of the common component(s) $h = 0$.

Example 2.4. We will study the ideal $(x^4 y, x^7 + xy^4) \subset \mathbb{C}[x, y]$. We take the generic curve $x^4 y + x^7 + xy^4$ of this ideal and we perform the same blow-ups as are used to principalize the ideal.

	principalization of $(x^4y, x^7 + xy^4)$	embedded resolution of $x^4y + x^7 + xy^4 = 0$
<u>Chart 1</u> (x, xy)	$x^5(y, x^2 + y^4)$	$x^5(y + x^2 + y^4)$
	$E_1 \leftrightarrow x = 0$	$E_1 \leftrightarrow x = 0$
<u>Chart 2</u> (xy, y)	$xy^5(x^3, x^6y^2 + 1)$	$xy^5(x^3 + x^6y^2 + 1)$
	$E \leftrightarrow x = 0$	$E \leftrightarrow x = 0$
	$E_1 \leftrightarrow y = 0$	$E_1 \leftrightarrow y = 0$
<u>Chart 1.1</u> (x, xy)	$x^6(y, x + x^3y^4)$	$x^6(y + x + x^3y^4)$
	$E_2 \leftrightarrow x = 0$	$E_2 \leftrightarrow x = 0$
<u>Chart 1.2</u> (xy, y)	$x^5y^6(1, x^2y + y^3)$	$x^5y^6(1 + x^2y + y^3)$
	$E_1 \leftrightarrow x = 0$	$E_1 \leftrightarrow x = 0$
	$E_2 \leftrightarrow y = 0$	$E_2 \leftrightarrow y = 0$
<u>Chart 1.1.1</u> (x, xy)	$x^7(y, 1 + x^6y^4)$	$x^7(y + 1 + x^6y^4)$
	$E_3 \leftrightarrow x = 0$	$E_3 \leftrightarrow x = 0$
<u>Chart 1.1.2</u> (xy, y)	$x^6y^7(1, x + x^3y^6)$	$x^6y^7(1 + x + x^3y^6)$
	$E_2 \leftrightarrow x = 0$	$E_2 \leftrightarrow x = 0$
	$E_3 \leftrightarrow y = 0$	$E_3 \leftrightarrow y = 0$

We can also construct the intersection diagram of this principalization and resolution, together with the numerical data (N, ν).
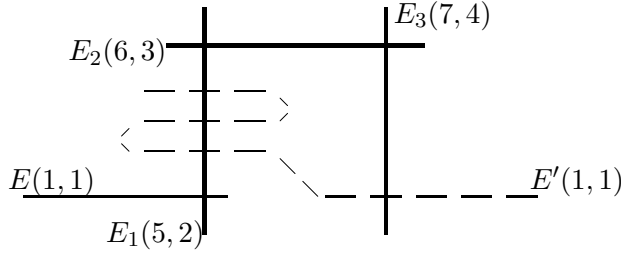
The curves E and E' are the components of the strict transform of the generic curve. The first one is also the support of the weak transform of the ideal, the second one does not occur in the principalization.

Remark 2.5. In this example you can also see that the numerical data of the principalization and those of the resolution are the same. This is true in general. The equality of the ν_i is obvious, the N_i are equal since for general $\lambda_1, \dots, \lambda_l$, the vanishing order of a divisor E along $\lambda_1 f_1 + \dots + \lambda_l f_l$ is equal to the minimum of the vanishing orders of E along the f_i.

Remark 2.6. Although the embedded resolution of the generic curve is in general not minimal, not every blow-up is allowed in the minimal principalization. We will only blow up with center on the intersection of at least one exceptional curve with the support of the weak transform of the ideal. Note

that this means that 'superfluous' blowing-ups in the non-minimal embedded resolution of our generic curve have center on the intersection of the exceptional locus with the strict transform of the generic curve.

3 Relations between numerical data

For the numerical data of an embedded resolution of a generic curve of the ideal $(f_1, \ldots, f_l) \subset \mathbb{C}[x, y]$, we know that the following relation holds: when $E(N, \nu)$ is an exceptional curve that intersects k times other components $E_i(N_i, \nu_i)$ and $\alpha_i = \nu_i - \frac{\nu}{N} N_i$ for $i = 1, \ldots, k$, then

$$\sum_{i=1}^{k} \alpha_i = k - 2.$$

This relation between the numerical data was proved by Loeser in [L] and generalized by the second author in [V2].

The intersection diagram with the numerical data of the principalization is almost the same as the one that arises from the (in general nonminimal) resolution of the generic curve $h \cdot (\lambda_1 f_1' + \ldots + \lambda_l f_l') = 0$. Here we use again the notation of the previous section, so we suppose that $(f_1, \ldots, f_l) = (h)(f_1', \ldots, f_l')$, with $(f_1', \ldots, f_l')$ a finitely supported ideal. The only difference between the two intersection diagrams is that the strict transform of $\lambda_1 f_1' + \ldots + \lambda_l f_l' = 0$ disappears in the principalization.

So we can divide the k intersections of an exceptional curve of an embedded resolution of the generic curve in two groups: there are n intersections with the strict transform of $\lambda_1 f_1' + \ldots + \lambda_l f_l' = 0$ and $m = k - n$ intersections that are preserved in the intersection diagram of the principalization of the ideal. Since we know that the first mentioned curve has numerical data $(1, 1)$, we can write -after renumbering the intersections- that

$$\sum_{i=1}^{m} \alpha_i + n(1 - \frac{\nu}{N}) = m + n - 2,$$

or

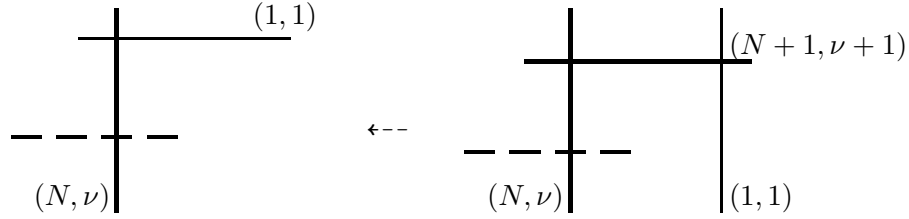$$\sum_{i=1}^{m} \alpha_i = m - 2 + \frac{\nu n}{N}. \tag{1}$$

Proposition 3.1. *Let $E(N, \nu)$ be an exceptional curve of a principalization of $(f_1, \ldots, f_l) \subset \mathbb{C}[x, y]$, intersecting $E_i(N_i, \nu_i)$ for $i = 1, \ldots, m$, and set $\alpha_i = \nu_i - \frac{\nu}{N} N_i$ for all $i \in \{1, \ldots, m\}$. Then $-1 \leq \alpha_i < 1$ for every i. Moreover, $\alpha_i = -1$ only occurs when $m = 1$.*

Proof. This proposition has been proven by Loeser in [L, Proposition II.3.1] for the numerical data of minimal embedded resolutions. Since we already

noticed that the numerical data of the principalization and the (possibly non-minimal) embedded resolution of a generic curve are the same (see Remark 2.5), we can look at these data as if they were coming from a resolution of the generic curve.

We can divide the exceptional curves in two groups: the ones that were first created are part of the minimal embedded resolution of the generic curve. As a consequence of the mentioned theorem of Loeser, the α_i that originate from these will satisfy the condition $-1 \leq \alpha_i < 1$. The second group of blow-ups will have center on the intersection of one exceptional curve and the strict transform of the generic curve. Moreover, since we suppose that we have already an embedded resolution, we know that the multiplicity of the generic curve in the center of the blow-up is one.

So we only need to look at the following situation:

We can suppose that $-1 \leq 1 - \frac{\nu}{N} < 1$ (or that $0 < \frac{\nu}{N} \leq 2$) and we only need to show that

(i) $-1 \leq \nu + 1 - \frac{\nu}{N}(N+1) < 1$,

(ii) $-1 \leq \nu - \frac{\nu+1}{N+1}N < 1$ and

(iii) $-1 \leq 1 - \frac{\nu+1}{N+1} < 1$.

This is straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.2. *Let $E(N,\nu)$ be an exceptional curve of a principalization σ of an ideal $\mathcal{I} \subset \mathbb{C}[x,y]$. Suppose that E intersects the other components $E_i(N_i,\nu_i)$ for $i = 1, \ldots, m$ of the total transform $\sigma^*\mathcal{I}$. Let $\alpha_i = \nu_i - \frac{\nu}{N}N_i$ for all $i \in \{1, \ldots, m\}$. Then we have the following statements.*

1. *At most one $E_i, 1 \leq i \leq m$, occurs such that $\alpha_i < 0$.*

2. *If $m \geq 3$, then there is at most one i such that $\alpha_i \leq 0$.*

3. *If $m = 2$, we see that $\frac{\nu_1}{N_1} < \frac{\nu}{N} \Rightarrow \frac{\nu}{N} < \frac{\nu_2}{N_2}$.*

This is a direct consequence of the previous proposition and equation (1). In [V3] there are almost the same results for the numerical data of an embedded resolution of a curve. However, the analogue of the third statement in that

context is an equivalence instead of an implication. Roughly said, this is due to the presence of the positive term $\frac{\nu n}{N}$ in our equation (1).

The mentioned corollary in [V3] is used there to determine the 'ordered tree'-structure of the resolution graph. The same can be done in our case. We can draw a dual principalization graph by associating a vertex to every exceptional curve and every (analytically irreducible) component of the support of the weak transform. For each intersection we have an edge, connecting the corresponding vertices.

By using Corollary 3.2, it is not so difficult to derive the next proposition. For example, this can be done as in [V3, Theorem 3.3].

Proposition 3.3. *The part of the dual principalization graph where $\frac{\nu}{N}$ is minimal, is connected. Moreover, when we follow a path that moves away from this minimal part, the ratio $\frac{\nu}{N}$ will strictly increase.*

4   Poles of a zeta function of an ideal

In this section we always consider ideals $\mathcal{I} \subset \mathbb{C}[x,y]$ with $0 \in \text{Supp}(\mathcal{I})$. Since we study the local topological zeta function associated to $\mathcal{I}$, we need in fact only a principalization of $\mathcal{I}$ in the neighbourhood of 0.

We know that the only possible poles of the topological zeta functions are rational numbers $-\frac{\nu}{N}$ with (N,ν) numerical data of components of the minimal principalization. We can see that the largest candidate pole plays a special role. The following arguments show that it is always a pole. If there are different components with this maximal ratio $-\frac{\nu}{N}$, these components need to intersect and we find a pole of order two. Moreover, this is the only value where a pole of order two is possible. This is a consequence of the 'ordered tree'-structure of the graph (see Proposition 3.3). When there is only one component $E(N,\nu)$ with this minimal ratio, we have a candidate pole of order one. Its residue is then given by

$$R = \frac{1}{N}\left(2 - m + \sum_{i=1}^{m}\frac{1}{\alpha_i}\right),$$

where we suppose that E intersects m times other components $E_i(N_i,\nu_i)$ $(i = 1,\ldots,m)$ of the principalization, and $\alpha_i = \nu_i - \frac{\nu}{N}N_i$. When $\frac{\nu}{N}$ is minimal, then $0 < \alpha_i < 1$ for every i, so $R > 0$ and $-\frac{\nu}{N}$ is a pole.

Not every other candidate pole gives rise to a pole. For the topological zeta function associated to a curve in $\mathbb{C}^2$, the second author proved the following theorem in [V3].

Theorem 4.1. *Let $f \in \mathbb{C}[x,y]$ be a non-constant polynomial satisfying $f(0) = 0$, and let $h : X \to \mathbb{C}^2$ be the minimal embedded resolution of $f^{-1}\{0\}$*

in a neighbourhood of 0. Let $E_i(N_i, \nu_i)$ *be the irreducible components of* $h^{-1}(f^{-1}\{0\})$ *with their associated numerical data. We have that* s_0 *is a pole of* $Z_{top,f}(s)$ *if and only if* $s_0 = -\frac{\nu_i}{N_i}$ *for some exceptional curve* E_i *intersecting at least three times other components or* $s_0 = -\frac{1}{N_i}$ *for some irreducible component* E_i *of the strict transform of* $f = 0$.

This gives a criterion to filter the poles out of the series of candidate poles. The next theorem will do the same for the topological zeta function associated to an ideal in $\mathbb{C}[x, y]$. With this theorem we can easily determine the poles of the zeta function when we have the principalization of the ideal.

Theorem 4.2. *Let* $\mathcal{I} \subset \mathbb{C}[x, y]$ *be an ideal satisfying* $0 \in Supp(\mathcal{I})$ *and* $\sigma : X \to \mathbb{C}^2$ *the minimal principalization of* $\mathcal{I}$ *in a neighbourhood of 0. Let* $E_\bullet(N_\bullet, \nu_\bullet)$ *be the components of the support of the total transform* $\sigma^* \mathcal{I}$ *with their associated numerical data.*

The rational number s_0 *is a pole of the local topological zeta function of* $\mathcal{I}$ *if and only if one of the following conditions is satisfied:*

1. $s_0 = -\frac{1}{N}$ *for a component* $E(N, \nu)$ *of the support of the weak transform of* $\mathcal{I}$;

2. $s_0 = -\frac{\nu}{N}$ *for* $E(N, \nu)$ *an exceptional curve that intersects no other component;*

3. $s_0 = -\frac{\nu}{N}$ *for* $E(N, \nu)$ *an exceptional curve that intersects once another component* $E_i(N_i, \nu_i)$ *with* $\nu_i - \frac{\nu}{N} N_i \neq -1$;

4. $s_0 = -\frac{\nu}{N}$ *for* $E(N, \nu)$ *an exceptional curve that intersects two times other components* $E_i(N_i, \nu_i)$ *and* $E_j(N_j, \nu_j)$ *with* $(\nu_i - \frac{\nu}{N} N_i) + (\nu_j - \frac{\nu}{N} N_j) \neq 0$;

5. $s_0 = -\frac{\nu}{N}$ *for* $E(N, \nu)$ *an exceptional curve that intersects at least three times other components.*

Remark 4.3. In the proof we will work with the following notation. If $E(N, \nu)$ is a curve in the support of the total transform of $\mathcal{I}$ that intersects once another curve $E_i(N_i, \nu_i)$, we write $\alpha = \nu_i - \frac{\nu}{N} N_i$. If $E(N, \nu)$ intersects the curves $E_{i_1}(N_{i_1}, \nu_{i_1}), E_{i_2}(N_{i_2}, \nu_{i_2}), \ldots, E_{i_m}(N_{i_m}, \nu_{i_m})$, we write $\alpha_j = \nu_{i_j} - \frac{\nu}{N} N_{i_j}$.

Proof. We have already said that the only possible pole of order two is the largest candidate pole. We can see that if s_0 is maximal, at least one of the five conditions is satisfied.

Now we will calculate the contribution to the residue of s_0 as a pole of order one in the various cases. (We can suppose that all $\alpha_i \neq 0$.) We will see that the five situations of the theorem are the only ones where that contribution is non-zero. Moreover we will show that this contribution

is negative, unless s_0 is the largest candidate pole. Notice that this last condition corresponds with "every $\alpha_i > 0$".

Suppose that $s_0 = -\frac{1}{N}$ for a component $E(N, \nu)$ of the support of the weak transform. Such a component only intersects one exceptional curve $E_i(N_i, \nu_i)$ and we see that the contribution to the residue of a pole of order one is $R = \frac{1}{N\alpha}$. This is positive if s_0 is the largest pole and negative otherwise.

If we have an exceptional curve $E(N, \nu)$ that doesn't intersect any other component, we know that this is the only curve in the principalization. So the topological zeta function is given by $\frac{2}{\nu + sN}$ and the value $s_0 = -\frac{\nu}{N}$ is a pole.

Let $s_0 = -\frac{\nu}{N}$ for $E(N, \nu)$ an exceptional curve that intersects once another component. The contribution to the residue for a pole of order one, is $R = \frac{1}{N}(2 - 1 + \frac{1}{\alpha})$. So we see immediately that

$$R = 0 \Leftrightarrow \alpha = -1.$$

The case $\alpha = 0$ is excluded, because we only look at candidate poles of order one. If $\alpha > 0$, then $R > 0$. If $\alpha < 0$ and $\alpha \neq -1$, we can use $\alpha > -1$ to conclude that $R < 0$.

Now suppose that $s_0 = -\frac{\nu}{N}$ for $E(N, \nu)$ an exceptional curve that intersects two times other components. In this case $R = \frac{1}{N}(2 - 2 + \frac{1}{\alpha_1} + \frac{1}{\alpha_2}) = \frac{1}{N}\frac{\alpha_1 + \alpha_2}{\alpha_1 \alpha_2}$. We conclude that

$$R = 0 \Leftrightarrow \alpha_1 + \alpha_2 = 0.$$

If $\alpha_1 + \alpha_2 \neq 0$, we can use equation (1) to know that $\alpha_1 + \alpha_2 > 0$. So we are only interested in the sign of $\alpha_1 \alpha_2$ to know the sign of R. We know that α_1 and α_2 can't be both negative. If they are both positive, then s_0 is the largest pole and $R > 0$. In the other case we have $R < 0$.

The next case is where $s_0 = -\frac{\nu}{N}$ for $E(N, \nu)$ an exceptional curve that intersects at least three times an other component. Here, the contribution to the residue $R = \frac{1}{N}(2 - m + \sum_{i=1}^{m} \frac{1}{\alpha_i})$ is always non-zero. If every $\alpha_i > 0$, we can easily conclude that $R > 0$. When there is a $\alpha_i < 0$, we can use the results for the resolution of a curve that are written in [V3, Proposition 2.8] to see that $R' := \frac{1}{N}\left(2 - (m + n) + \sum_{i=1}^{m} \frac{1}{\alpha_i} + \frac{n}{1 - \frac{\nu}{N}}\right) < 0$. (Here we use the notation of the previous section.) Because we know that there can exist at most one negative α, we can deduce that $0 < 1 - \frac{\nu}{N} < 1$ and

$$R = R' + \frac{1}{N}\left(n - \frac{n}{1 - \frac{\nu}{N}}\right) < 0.$$

From these calculations we can conclude that all contributions to the residue are negative if s_0 is not maximal. So if one of these contributions is non-zero, the total residue is non-zero and s_0 is a pole of order one. $\qquad\square$

12

Remark 4.4. When we work with one element $f \in \mathbb{C}[x, y]$ instead of an ideal, only the first and the last case can occur. Moreover, a principalization of the ideal (f) is the same as an embedded resolution of the curve given by $f = 0$. So this theorem is a generalization of Theorem 4.1.

Example 4.5. We will continue Example 2.4 and calculate explicitly the topological zeta function of the ideal $\mathbf{f} = (x^4 y, x^7 + xy^4) \subset \mathbb{C}[x, y]$. With the calculations done in the previous example, we can see that

$$Z_{top,\mathbf{f}}(s) = \frac{1}{4 + 7s} + \frac{1}{(3 + 6s)(4 + 7s)} + \frac{1}{(2 + 5s)(3 + 6s)} + \frac{1}{(1 + s)(2 + 5s)}.$$

With a little calculation, we can simplify this expression to

$$Z_{top,\mathbf{f}}(s) = \frac{5s^2 + 16s + 8}{(4 + 7s)(2 + 5s)(1 + s)},$$

which implies that the poles of this function are $-4/7, -2/5$ and -1.

We can obtain the same result by using Theorem 4.2 in the following way:

- $E(1, 1)$ is a component of the support of the weak transform, so -1 is a pole;

- $E_1(5, 2)$ intersects twice other components, with $(1 - \frac{2}{5}) + (3 - \frac{2}{5}6) \neq 0$, so $-2/5$ is a pole;

- $E_2(6, 3)$ also has two intersections with other components, this time with $(4 - \frac{3}{6}7) + (2 - \frac{3}{6}5) = 0$, hence the candidate pole $-1/2$ is no pole;

- $E_3(7, 4)$ intersects one other component with $3 - \frac{4}{7}6 \neq -1$, so this gives the last pole $-4/7$.

5   Determination of all possible poles

In this section, we will determine which numbers can occur as a pole of a topological zeta function associated to an ideal in dimension 2. For the topological zeta function of a curve, this question has been answered in [SV] and [LSV].

In the first article, Segers and the second author proved that the poles smaller than $-\frac{1}{2}$ are given by $\{-\frac{1}{2} - \frac{1}{i} \mid i \in \mathbb{Z}_{>1}\}$. In the second article, Lemahieu, Segers and the second author showed that every rational number in the interval $[-\frac{1}{2}, 0)$ is a pole of a zeta function of a curve. This determines all possible poles.

We will prove an analogue of these results for the topological zeta function of an ideal in dimension 2.

For an exceptional variety $E(N, \nu)$ of the minimal embedded resolution of a curve, one can show that $\nu \leq N$. Analogously, we prove the following proposition.

Proposition 5.1. *Let $E(N, \nu)$ be an exceptional curve of the minimal principalization of an ideal in $\mathbb{C}[x, y]$. Then the numerical data satisfy*

$$\nu \leq N + 1.$$

Proof. We will prove this proposition by induction. If $E_1(N_1, \nu_1)$ is the first created exceptional curve, then $\nu_1 = 2$ and $N_1 \geq 1$, so the statement is proven. Now we will suppose that the inequality is satisfied for all already created exceptional curves and we will prove it for the next one.

- First, suppose that the center of the blow-up is contained in two exceptional curves $E_{i_1}(N_{i_1}, \nu_{i_1})$ and $E_{i_2}(N_{i_2}, \nu_{i_2})$. Then we see that $\nu_i = \nu_{i_1} + \nu_{i_2}$ and $N_i = N_{i_1} + N_{i_2} +$ (minimal multiplicity of the generators of the ideal in the center). If we use the induction hypothesis, we see that $\nu_i \leq N_{i_1} + N_{i_2} + 2$, but we also know that $N_i \geq N_{i_1} + N_{i_2} + 1$, so $\nu_i \leq N_i + 1$.

- When only E_{i_1} exists, then $\nu_i = \nu_{i_1} + 1 \leq N_{i_1} + 2$ and $N_i = N_{i_1} +$ (minimal multiplicity of the generators of the ideal in the center) $\geq N_{i_1} + 1$, so $\nu_i \leq N_i + 1$. $\square$

Remark 5.2. This implies that all candidate poles of the topological zeta function associated to an ideal are rational elements of $[-1, 0) \cup \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>0}\}$. The next proposition will show that every rational number in this range really occurs as a pole of a certain topological zeta function. Hence we have a complete description of the possible poles.
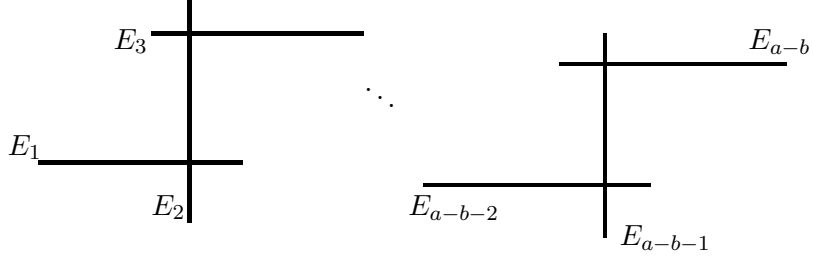
Theorem 5.3. *The set of rational numbers s_0 for which there exists an ideal $\mathcal{I} \subset \mathbb{C}[x, y]$ such that $Z_{top, \mathcal{I}}(s)$ has a pole in s_0, is given by $\mathbb{Q} \cap ([-1, 0) \cup \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>0}\})$.*

Proof. Choose $a, b \in \mathbb{Z}_{\geq 0}$ with $a > b$. Look at the ideal

$$(x^b y , \ x^a + y^{b+1}) \subset \mathbb{C}[x, y].$$

After principalization, we find the following numerical data and intersection diagram:

$$E_1(b + 1, 2) , \ E_2(b + 2, 3) , \ \ldots , \ E_{a-b-1}(a - 1, a - b) , \ E_{a-b}(a, a - b + 1).$$

The last exceptional variety only intersects E_{a-b-1} with $\alpha \neq -1$, so this causes a pole in $-\frac{a-b+1}{a}$. Easy calculations show that this implies that every element of $\mathbb{Q} \cap ([-1, 0) \cup \{-1 - \frac{1}{i} \mid i \in \mathbb{Z}_{>0}\})$ occurs as a pole of the topological zeta function of an ideal in $\mathbb{C}[x, y]$. $\qquad\square$

6   Other zeta functions

There are finer variants of the topological zeta function of an ideal. For instance, using the notation of the introduction, there is the (local) *Hodge zeta function*

$$Z_{Hod,\mathbf{f}}(s) = \sum_{I \subset T} H(E_I^{\circ} \cap \sigma^{-1}(0); u, v) \prod_{i \in I} \frac{uv - 1}{(uv)^{\nu_i + sN_i} - 1} \in \mathbb{Q}(u, v)((uv)^{-s})$$

for the ideal $\mathbf{f}$, where $H(\,\cdot\,; u, v) \in \mathbb{Z}[u, v]$ denotes the Hodge polynomial. Even finer is the (local) motivic zeta function of $\mathbf{f}$, which was already mentioned in the introduction. Its formula involves classes in the Grothendieck ring of algebraic varieties, instead of Euler characteristics or Hodge polynomials. We refer to e.g. [DL2], [R] or [V4] for these zeta functions and their global versions associated to one polynomial and to [VZ] for ideals.

The results in this paper on poles of the topological zeta function of an ideal in dimension 2, i.e. Theorems 4.2 and 5.3, are also valid for the Hodge and the motivic zeta function. We chose not to give the details here about these zeta functions, since for results of the kind we proved, the version for the topological zeta function is the strongest, and implies the same results for the finer zeta functions.

The point is that the motivic zeta function specializes to the Hodge zeta function, which in turn specializes to the topological zeta function. (Note for instance that $H(\,\cdot\,; 1, 1) = \chi(\cdot)$.) In particular, a pole of the topological zeta function will induce a pole of the other two. (The converse is not clear.) We refer to [R] and [RV] for the precise description of the notion of a pole for the Hodge and the motivic zeta function. Here we should note that the analogue of Theorem 4.2 for the finer zeta functions also requires the verification of the following in the context of for instance Hodge polynomials. Exceptional curves intersecting once or twice other components such that $\alpha = -1$ or $\alpha_1 + \alpha_2 = 0$, respectively, should not contribute to the residue

of the induced candidate pole. Now this is as straightforward as with Euler characteristics (and well known).

Theorems 4.2 and 5.3 are also valid for (most) p-adic Igusa zeta functions. We briefly introduce the necessary notation to introduce these zeta functions and to state the precise result.

Let K be a finite extension of the p-adic numbers with valuation ring R, maximal ideal P, and residue field $\bar{K} = R/P(\cong \mathbb{F}_q)$. Denote for $z \in K$ by $|z|$ its standard absolute value, and put $\|z\| := \max_{1 \leq i \leq l} |z_i|$ for $z = (z_1, \ldots, z_l) \in K^l$. Let $f_1, \ldots, f_l$ be polynomials in $K[x_1, \ldots, x_n]$. The (local) p-adic Igusa zeta function associated to the mapping $\mathbf{f} = (f_1, \ldots, f_l) : K^n \to K^l$ is

$$Z_{K,\mathbf{f}}(s) := \int_{P^n} \|\mathbf{f}(x)\|^s \, |dx|$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$, where $|dx|$ is the usual Haar measure on K^n. A global version consists in replacing P^n by R^n. This function is analytic in s and admits a meromorphic continuation to $\mathbb{C}$ as a rational function of q^{-s}. This was first proved by Igusa for $l = 1$ (see [I]). For arbitrary l there are different proofs in [M], [D1] and [VZ].

Considering polynomials $f_1, \ldots, f_l$ over a number field F, one can study $Z_{K,F}(s)$ for all (non-archimedean) completions K of F. For all but finitely many completions K there is a concrete formula for $Z_{K,F}(s)$ in terms of a principalization of the ideal $(f_1, \ldots, f_l)$, similar to the formulas for the other zeta functions in this paper. This was proved for $l = 1$ by Denef in [D2], and can be generalized to arbitrary l, see [VZ, (2.3)]. (In fact the motivic zeta function specializes to 'almost all' p-adic zeta functions, see [DL2, (2.4)].)

When the number field F is large enough, then for all but finitely many completions K of F we have that the analogues of Theorems 4.2 and 5.3 are valid for $Z_{K,F}(s)$, replacing 'pole' by 'real pole'. One can derive this from the results for the topological zeta function, or by completely analogous proofs. (Previous such results for $l = 1$ are in [V1] and [S].)

References

[AKMW] D. Abramovich, K. Karu, K. Matsuki, J. Włodarczyk, *Torification and factorization of birational maps,* J. Amer. Math. Soc. **15** (2002), 531-572.

[D1] J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety,* Invent. Math. **77** (1984), 1-23.

[D2] J. Denef, *On the degree of Igusa's local zeta function,* Amer. J. Math. **109** (1987), 991-1008.

[DL1] J. Denef and F. Loeser, *Caractéristiques d' Euler-Poincaré, fonctions zêta locales, et modifications analytiques,* J. Amer. Math. Soc. **5** (1992), 705-720.

[DL2] J. Denef and F. Loeser, *Motivic Igusa zeta functions,* Journal of Algebraic Geometry **7** (1998), 505-537.

[H]   H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero,* Ann. Math. **79** (1964), 109-326.

[I]   J. Igusa, *Complex powers and asymptotic expansions I,* J. Reine Angew. Math. **268/269** (1974), 110-130; *II*, ibid. **278/279** (1975), 307-321.

[J]   J.-P. Jouanolou, *Théorèmes de Bertini et applications,* Progr. Math. **42**, Birkhäuser, Boston, 1983.

[L]   F. Loeser, *Fonctions d' Igusa p-adiques et polynômes de Bernstein,* Amer. J. Math. **110** (1988), 1-21.

[LSV]   A. Lemahieu, D. Segers and W. Veys, *On the poles of topological zeta functions,* Proc. Amer. Math. Soc. (to appear).

[M]   D. Meuser, *On the rationality of certain generating functions,* Math. Ann. **256** (1981), 303-310.

[R]   B. Rodrigues, *On the geometric determination of the poles of Hodge and motivic zeta functions,* J. Reine Angew. Math. **578** (2005), 129-146.

[RV]   B. Rodrigues and W. Veys, *Poles of zeta functions on normal surfaces,* Proc. London Math. Soc. **87** (2003), 164-196.

[S]   D. Segers, *On the smallest poles of Igusa's p-adic zeta functions,* Math. Z. **252** (2006), 429-455.

[SV]   D. Segers and W. Veys, *On the smallest poles of topological zeta functions,* Compositio Math. **140** (2004), 130-144.

[V1]   W. Veys, *On the poles of Igusa zeta functions for curves,* J. Lond. Math. Soc., **41** (1990), 27-32.

[V2]   W. Veys, *Relations between numerical data of an embedded resolution,* Amer. J. Math. **113** (1991), 573-592.

[V3]   W. Veys, *Determination of the poles of the topological zeta function for curves,* Manuscripta Math. **87** (1995), 435-448.

[V4]   W. Veys, *Arc spaces, motivic integration and stringy invariants,* Advanced Studies in Pure Mathematics, Proceedings of "Singularity Theory and its applications, Sapporo (Japan), 16-25 September 2003" (to appear), 43p.

[VZ]   W. Veys and W.A. Zuniga-Galindo, *Zeta functions for analytic mappings, log-principalization of ideals, and Newton polyhedra,* preprint (2006), 23p.